The infinite volume limit in generalized mean field disordered models

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Abstract

We generalize the strategy we recently introduced to prove the existence of the thermodynamic limit for the Sherrington-Kirkpatrick and p-spin models, to a wider class of mean field spin glass systems, including models with multi-component and non-Ising type spins, mean field spin glasses with an additional Curie-Weiss interaction, and systems consisting of several replicas of the spin glass model, where replicas are coupled with terms depending on the mutual overlaps.

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1 Introduction

In a recent paper [1], we introduced a simple interpolation method, which allows to prove the existence of the thermodynamic limit for the quenched average of the free energy and ground state energy per site, for a wide class of mean field spin glass models [2]. This class includes, for instance, the well known Sherrington-Kirkpatrick (SK) model [3, 4] and Derrida's p-spin model [5, 6, 7], for even p. Moreover, we proved almost sure convergence, with respect to the quenched disorder, without taking the average. Subsequently, this strategy has been developed in [8] to include, among the others, the REM [5] and GREM [9], and in [10], where finite connectivity models are considered. In the present paper, we generalize the strategy of [1] in a different direction. The class of models for which we provide a proof of the existence of the thermodynamic limit embraces, for instance, models where the spin degrees of freedom σ_i have several components, which are not necessarily two-valued Ising variables. In the same way, we show how to treat the case where a Curie-Weiss interaction term is added to the mean field spin glass Hamiltonian. Finally, the same results hold for a system composed of several replicas (i.e., identical copies with the same disorder realization) of the mean field spin glass model, where replicas are coupled together by an interaction term, which depends on their mutual overlaps. In all of these cases, when the method of [1] is naively applied, there appear terms which spoil the simple subadditivity argument which works for the SK and p-spin models. The main purpose of this paper is to show that the effect of these potentially dangerous terms can be eliminated, by suitably decomposing the configuration space. This idea was introduced by Michel Talagrand in [11], and developed in a series of important applications. Among these, Talagrand proposed a very interesting generalization of the broken replica bounds [12] to the case of systems made of two coupled replicas.

The organization of the paper is as follows. In Section 2, we introduce some basic definitions concerning mean field spin glass models. In Section 3, we state the main results of the paper, and we give some physically meaningful examples of models to which they can be applied. Section 4 contains the proof of the results. Finally, Section 5 is dedicated to conclusions.

2 Basic definitions

In this Section, we recall some basic definitions concerning mean field spin glasses, without making reference to any specific model.

The generic configuration σ of the system is defined by N spin degrees

of freedom $\sigma_1, \sigma_2, \ldots$. We suppose each σ_i to belong to a set $\mathcal{S} \in \mathbb{R}^n$, $n \in \mathbb{N}$, equipped with an *a priori* measure ν. For instance, the case $\mathcal{S} = \{-1, +1\}$ and $\nu = 1/2(\delta_{-1} + \delta_{+1})$ corresponds to the usual Ising two-valued variables. The Hamiltonian of the model, $H_N(\sigma, J)$, depends on the spin configuration, on the system size N and on some quenched disorder, which we denote as J. Of course, the Hamiltonian can also depend on some additional external fields, e.g., on the magnetic field h. The mean field character of the model consists in the condition that, if two configurations σ and σ' are connected by a permutation of the site indices, the random variables $H_N(\sigma, J)$ and $H_N(\sigma', J)$ have the same distribution.

For a given inverse temperature β, we introduce the disorder dependent partition function $Z_N(\beta, J)$, the quenched average of the free energy per site $f_N(\beta)$, and the Boltzmann-Gibbs state ω_J, according to the definitions

$$Z_N(\beta, J) = \int_{\mathcal{S}^N} d\nu(\sigma_1) \ldots d\nu(\sigma_N) \exp(-\beta H_N(\sigma, J)), \tag{1}$$

$$-\beta f_N(\beta) = N^{-1} E \log Z_N(\beta, J), \tag{2}$$

$$\omega_J(A) = Z_N(\beta, J)^{-1} \int_{\mathcal{S}^N} d\nu(\sigma_1) \ldots d\nu(\sigma_N) A(\sigma) \exp(-\beta H_N(\sigma, J)), \tag{3}$$

where A is a generic function of the σ's.

Let us now introduce the important concept of replicas. Consider a generic number n of independent copies of the system, characterized by the spin variables $\sigma_i^{(1)}$, $\sigma_i^{(2)}$, ..., distributed according to the product of Boltzmann-Gibbs states

$$\Omega_J = \omega_J^{(1)} \omega_J^{(2)} \ldots \omega_J^{(n)},$$

where each $\omega_J^{(\alpha)}$ acts on the corresponding set of $\sigma_i^{(\alpha)}$'s, and all replicas are subject to the *same* sample J of the quenched disorder. For a generic smooth function F of the configuration of s replicas, we define the $\langle . \rangle$ averages as

$$\langle F(\sigma^{(1)}, \sigma^{(2)}, \ldots) \rangle = E \Omega_J(F(\sigma^{(1)}, \sigma^{(2)}, \ldots)),$$

where the Boltzmann-Gibbs averages Ω_J acts on the replicated σ variables, and E is the average with respect to the disorder J.

3 The existence of the thermodynamic limit

The main object of interest in the theory is the quenched free energy $f_N(\beta)$. First of all, one would like to prove that it admits a well defined limit, for $N \to \infty$, independently from the explicit calculation of the limit itself.

We restrict our analysis to the case of Gaussian models, i.e., models for which the $H_N(\sigma, J)$ are (correlated) Gaussian random variables. Of course, these random variables are fully characterized by their mean values

$$b_N(\sigma) = EH_N(\sigma, J)$$

and covariance matrix

$$c_N(\sigma, \sigma') = E(H_N(\sigma, J)H_N(\sigma', J)) - E(H_N(\sigma, J))E(H_N(\sigma', J)).$$

In order to prove the existence of the thermodynamic limit, we suppose that the following conditions are satisfied. First of all, we require that

$$\frac{b_N(\sigma)}{N} = g(m_N^{(1)}(\sigma), \ldots, m_N^{(k)}(\sigma)) + O(N^{-1}). \tag{4}$$

Here, $k \in \mathbb{N}$, g is a smooth function of class C^1 and the $m_N^{(i)}(\sigma)$ are bounded functions, with $Nm_N^{(i)}$ additive in the system size. In other words,

$$|m_N^{(i)}(\sigma)| \leq M \qquad \forall i, N, \sigma \tag{5}$$

and

$$Nm_N^{(i)}(\sigma) = N_1 m_{N_1}^{(i)}(\sigma^{(1)}) + N_2 m_{N_2}^{(i)}(\sigma^{(2)}), \tag{6}$$

if $N = N_1 + N_2$ and if the configuration σ can be decomposed as

$$\sigma = (\sigma_1^{(1)}, \ldots, \sigma_{N_1}^{(1)}, \sigma_1^{(2)}, \ldots, \sigma_{N_2}^{(2)}).$$

As for the covariance matrix, we require that

$$\frac{c_N(\sigma, \sigma')}{N} = f(Q_N^{(1)}(\sigma, \sigma'), \ldots, Q_N^{(k)}(\sigma, \sigma')) + O(N^{-1}), \tag{7}$$

where f is a convex function with continuous derivatives. The variables $Q_N^{(i)}$ must satisfy properties analogous to (5)-(6), i.e.,

$$|Q_N^{(i)}(\sigma, \sigma)| \leq M \qquad \forall i, N, \sigma \tag{8}$$

and

$$NQ_N^{(i)}(\sigma, \sigma') = N_1 Q_{N_1}^{(i)}(\sigma^{(1)}, \sigma'^{(1)}) + N_2 Q_{N_2}^{(i)}(\sigma^{(2)}, \sigma'^{(2)}). \tag{9}$$

It is interesting to notice that the models considered in [1] have the additional properties that $c_N(\sigma, \sigma)$ does not depend on the configuration σ, and that g is a linear function.

Now, we can state our result:

Theorem 1. *If conditions (4) to (9) are satisfied, then the thermodynamic limit of the quenched free energy exists:*

$$\lim_{N\to\infty} -\frac{1}{N\beta} E \ln Z_N(\beta) = f(\beta). \tag{10}$$

Moreover, the disorder dependent free energy converges almost surely, with respect to the disorder realization:

$$\lim_{N\to\infty} -\frac{1}{N\beta} \ln Z_N(\beta) = f(\beta) \qquad J - \textit{almost surely}, \tag{11}$$

and its disorder fluctuations can be estimated as

$$P\left(\left|-\frac{1}{N\beta} \ln Z_N(\beta) - f_N(\beta)\right| \geq u\right) \leq 2\exp\left(-\frac{Nu^2}{2L}\right), \tag{12}$$

where

$$L = \max_{|x_i|\leq M\ \forall i} |f(x_1, \ldots, x_k)|, \tag{13}$$

and M is the same constant as in (8).

Remark As explained in [1], from Eqs. (10)-(12) follows also the convergence, both under quenched average and J-almost surely, of the ground state energy per site.

Before we turn to the proof of the Theorem, we give a few examples of physically meaningful systems to which it applies.

1. The SK model with non-Ising type spins, defined as

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq N} J_{ij}\sigma_i\sigma_j - h\sum_{i=1}^{N} \sigma_i. \tag{14}$$

Here, and in the following examples, the couplings J_{ij} are independent identically distributed Gaussian random variables, with mean zero and unit variance, while h is the magnetic field. As for the spin degrees of freedom, we suppose that $\sigma_i \in \mathcal{S} = [-a, a]$, while the measure ν on $\mathcal{S}$, which appears in the definition of the partition function, is arbitrary. In this case,

$$\frac{b_N(\sigma)}{N} = -h\, m_N(\sigma) = -\frac{h}{N}\sum_{i=1}^{N} \sigma_i$$

and conditions (4) to (6) are clearly satisfied, since $|m_N(\sigma)| \leq a$ and the total magnetization $\sum_i \sigma_i$ is linear in the system size. Of course, the function g in (4) is just $g(x) = -h\,x$. As regards the covariance matrix, one finds easily

$$\frac{c_N(\sigma, \sigma')}{N} = \frac{q^2_{\sigma\sigma'}}{2} + O(N^{-1}),$$

where

$$q_{\sigma\sigma'} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\sigma'_i, \qquad |q_{\sigma\sigma'}| \le a^2,$$

is the overlap of the two configurations. Since $Nq_{\sigma\sigma'}$ is additive and $f(x) = x^2/2$ is convex, conditions (7) to (9) are also satisfied.

2. The SK model with an additional Curie-Weiss interaction, defined as

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N}}\sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j - \frac{J_0}{N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i,$$

where J_0 is a non random constant and, again, $\sigma_i \in [-a, a]$. This model can be obtained from the previous one, if one supposes that the Gaussian variables J_{ij} in (14) have mean value $2J_0/\sqrt{N}$. This case can be dealt with in analogy with the previous one, with the only difference that

$$\frac{b_N(\sigma)}{N} = -J_0\, m_N(\sigma)^2 - h\, m_N(\sigma),$$

so that $g(x) = -h\,x - J_0\,x^2$.

3. The multi-replica SK model, with coupled replicas. In this case, the Hamiltonian depends on the configurations $\sigma^{(1)}, \ldots, \sigma^{(n)}$ of the n replicas, which interact through a term depending on the mutual overlaps:

$$H_N(\sigma^{(1)}, \ldots, \sigma^{(n)}, J) = -\frac{1}{\sqrt{N}}\sum_{1\le i<j\le N} J_{ij}(\sigma_i^{(1)}\sigma_j^{(1)} + \ldots + \sigma_i^{(n)}\sigma_j^{(n)}) + Ng(\{q_{ab}\}),$$

where g is a smooth C^1 function of all the overlaps. The check of properties (4) to (9) is trivial, and is left to the reader.

4. The SK model with Heisenberg type interaction, defined by the Hamiltonian

$$H_N(\sigma, J) = -\frac{1}{\sqrt{N}}\sum_{1\le i<j\le N} J_{ij}\vec{\sigma}_i\,\vec{\sigma}_j - \sum_{i=1}^{N}\vec{h}\,\vec{\sigma}_i, \tag{15}$$

where $\vec{\sigma}_i$ has n bounded components $\sigma_i^{(1)}, \ldots, \sigma_i^{(n)}$, and $\vec{u}\,\vec{v}$ denotes scalar product in $\mathbb{R}^n$. In this case,

$$\frac{c_N(\sigma, \sigma')}{N} = \frac{1}{2}\sum_{a,b=1}^{n}(q_{\sigma\sigma'}^{ab})^2 + O(N^{-1}), \tag{16}$$

where

$$q_{\sigma\sigma'}^{ab} = \frac{1}{N}\sum_{i=1}^{N}\sigma_i^{(a)}\sigma_i'^{(b)}.$$

It is instructive to verify explicitly that, for these models, the method introduced in [1] does not work, and requires an extension.

4 Proof of Theorem 1

We start with the proof of (12). Results of this kind were firstly obtained in [13] in the general context of the norms of Gaussian sample functions, and later developed in [14], and [15]. The book by Talagrand [11] demonstrates at length the usefulness of this idea in the applications to mean field spin glass theory. For later convenience, we give a selfcontained proof of the more general inequality

$$P\left(\left|-\frac{1}{N\beta}\ln Z_N^A(\beta)+\frac{1}{N\beta}E\ln Z_N^A(\beta)\right|\geq u\right)\leq 2\exp\left(-\frac{Nu^2}{2L}\right), \tag{17}$$

where

$$Z_N^A(\beta,J)=\int_{A_N}d\nu(\sigma_1)\dots d\nu(\sigma_N)\exp(-\beta H_N(\sigma,J)) \tag{18}$$

is a modified disorder-dependent partition function, with the sum over configurations restricted to an arbitrary nonrandom set A_N in the configuration space $\mathcal{S}^N$.

The restriction to a subset of the space of configurations has been exploited also in [11] in the case of the p-spin models.

We rewrite the Gaussian variables $H_N(\sigma,J)$ as

$$H_N(\sigma,J)=\xi_N(\sigma)+b_N(\sigma), \tag{19}$$

where, of course, $\xi_N(\sigma)$ is a centered Gaussian random variable, and

$$E(\xi_N(\sigma)\xi_N(\sigma'))=c_N(\sigma,\sigma').$$

Given $s\in\mathbb{R}$, we define

$$\varphi_N(t)=\ln E_1 G_N(t)=\ln E_1\exp\left(s\beta^{-1}E_2\ln Z_N^A(t)\right), \tag{20}$$

where the interpolating parameter t varies between 0 and 1, and $Z_N^A(t)$ is the auxiliary partition function

$$\begin{aligned} Z_N^A(t) &= Z_N^A(t,J_1,J_2,\beta) \qquad (21)\\ &= \int_{A_N}d\tilde{\nu}(\sigma)\exp(-\beta\sqrt{t}\xi_N^1(\sigma)-\beta\sqrt{1-t}\xi_N^2(\sigma)-\beta b_N(\sigma)). \end{aligned}$$

Here, $\xi_N^1(\sigma)$ and $\xi_N^2(\sigma)$ are two *independent* copies of the random variable $\xi_N(\sigma)$, with the same distribution, and E_1,E_2 denote average with respect to ξ^1 and ξ^2, respectively. For simplicity of notation, we set

$$d\tilde{\nu}(\sigma)=d\nu(\sigma_1)\dots d\nu(\sigma_N)$$

the *a priori* measure on $\mathcal{S}^N$, and

$$p_N(t,\sigma) = \frac{\exp(-\beta\sqrt{t}\xi_N^1(\sigma) - \beta\sqrt{1-t}\xi_N^2(\sigma) - \beta b_N(\sigma))}{Z_N^A(t)}$$

the modified Boltzmann weight.

It is very simple to check that

$$\varphi_N(1) - \varphi_N(0) = \ln E \exp\, s\beta^{-1}\left(\ln Z_N^A(\beta) - E\ln Z_N^A(\beta)\right). \tag{22}$$

As for the t derivative of $\varphi_N(t)$, an application of the formula

$$Ex_iF(\{x\}) = \sum_j E(x_ix_j)E\partial_{x_j}F(\{x\}) \tag{23}$$

which holds for any family of Gaussian random variables $\{x_i\}$ and any smooth function F, gives

$$\varphi_N'(t) \;=\; \frac{s^2}{2E_1G_N(t)}E_1\left\{G_N(t)\int_{A_N\times A_N} d\tilde{\nu}(\sigma)d\tilde{\nu}(\sigma')c_N(\sigma,\sigma')E_2p_N(t,\sigma)E_2p_N(t,\sigma')\right\}.$$

Thanks to the bound (8), one has

$$|\varphi_N'(t)| \le \frac{s^2}{2}\max_{\sigma,\sigma'}|c_N(\sigma,\sigma')| = \frac{Ns^2}{2}\max_{\sigma,\sigma'}|f(Q_N(\sigma,\sigma'))| = \frac{Ns^2L}{2}. \tag{24}$$

Therefore, using Eq. (22) and the obvious inequality

$$e^{|x|} \le e^x + e^{-x},$$

one finds

$$E\exp\left(N|s|\left|\frac{1}{N\beta}\ln Z_N^A(\beta) - \frac{1}{N\beta}E\ln Z_N^A(\beta)\right|\right) \le 2\exp\left(\frac{s^2NL}{2}\right). \tag{25}$$

By Tchebyshev's inequality,

$$P\left(\left|\frac{1}{N\beta}\ln Z_N^A(\beta) - \frac{1}{N\beta}E\ln Z_N^A(\beta)\right| \ge u\right) \le 2\exp\left(-N|s|u + \frac{s^2NL}{2}\right) \tag{26}$$

and, choosing $|s| = u/L$, one finally obtains the estimate (17). $\square$

Now, we can prove the main statements of Theorem 1, concerning the existence of the thermodynamic limit. For simplicity, we assume that

$$N^{-1}b_N(\sigma) = g(m_N(\sigma)) + O(N^{-1}),$$

and

$$N^{-1}c_N(\sigma,\sigma') = f(Q_N(\sigma,\sigma')) + O(N^{-1}),$$

corresponding to the case $k=1$ in Eqs. (4), (7), and that $L=1$ in (13). The general case can be obtained as a simple extension.

First of all, we prove the existence of the limit along sequences of the type $\{N_K\} = \{N_0 n^K\}$, with $n, N_0 \in \mathbb{N}$. As in [1], the idea is to find a suitable interpolation between the original system, of size N_K, and a system composed of n non interacting subsystems, of size N_{K-1} each. However, in the present case, it is also necessary to divide the configuration space into sets, such that $m_N(\sigma)$ and $Q_N(\sigma,\sigma)$ are approximately constant within each set. The idea of restricting to the set of configurations with given overlap was introduced by Michel Talagrand [11], and exploited in a series of important applications. For any $0<\varepsilon<1$, we can write

$$Z_{N_K}(\beta) = \sum_{i,j=0}^{[1/\varepsilon]} Z_{N_K}^{(ij)}(\beta) \equiv \sum_{i,j=0}^{[1/\varepsilon]} \int_{C_{ij}} d\tilde{\nu}(\sigma)\exp(-\beta H_{N_K}(\sigma,J)), \tag{27}$$

where

$$C_{ij} = \{\sigma\in\mathcal{S}^{N_K} : i\varepsilon \le Q_{N_K}(\sigma,\sigma), < (i+1)\varepsilon, j\varepsilon \le m_{N_K}(\sigma), < (j+1)\varepsilon\} \tag{28}$$

and $[x]$ denotes the integer part of x. Since $N_K = nN_{K-1}$, we can divide the system into n subsystems of N_{K-1} spins each, and we denote the configuration of the ℓ-th subsystem as $\sigma^{(\ell)}$, $\ell = 1,2,\ldots,n$. Of course, the following inequality holds

$$Z_{N_K}^{(ij)}(\beta) \ge \tilde{Z}_{N_K}^{(ij)}(\beta) = \int_{\tilde{C}_{ij}} d\tilde{\nu}(\sigma)\exp(-\beta H_{N_K}(\sigma,J)), \tag{29}$$

where

$$\begin{aligned} C_{ij} \supseteq \tilde{C}_{ij} \;\;=\;\; & \{\sigma\in\mathcal{S}^{N_K} : i\varepsilon \le Q_{N_{K-1}}(\sigma^{(\ell)},\sigma^{(\ell)}) < (i+1)\varepsilon, \\ & \qquad j\varepsilon \le m_{N_{K-1}}(\sigma^{(\ell)}) < (j+1)\varepsilon,\ \forall \ell\}. \end{aligned} \tag{30}$$

Now, we introduce an interpolating parameter $0\le t\le 1$, and the auxiliary partition function

$$\begin{aligned} \tilde{Z}_{N_K}^{(ij)}(t,\beta) \;\;=\;\; & \int_{\tilde{C}_{ij}} d\tilde{\nu}(\sigma)\exp\beta\left(-\sqrt{t}\xi_{N_K}(\sigma) - tb_{N_K}(\sigma) - \sqrt{1-t}\sum_{\ell=1}^{n}\xi_{N_{K-1}}^{\ell}(\sigma^{(\ell)})\right. \\ & \left. -(1-t)\sum_{\ell=1}^{n} b_{N_{K-1}}(\sigma^{(\ell)})\right), \end{aligned}$$

where the $\xi_N^\ell(\sigma)$ are n *independent* copies of the random variable $\xi_N(\sigma)$. Clearly, for the boundary values of the parameter t one has

$$-\frac{1}{N_K\beta}E\ln\tilde{Z}_{N_K}^{(ij)}(0,\beta)=-\frac{1}{N_{K-1}\beta}E\ln Z_{N_{K-1}}^{(ij)}(\beta) \tag{31}$$

and

$$-\frac{1}{N_K\beta}E\ln\tilde{Z}_{N_K}^{(ij)}(1,\beta)=-\frac{1}{N_K\beta}E\ln\tilde{Z}_{N_K}^{(ij)}(\beta)\geq-\frac{1}{N_K\beta}E\ln Z_{N_K}^{(ij)}(\beta). \tag{32}$$

As regards the t derivative, we apply the integration by parts formula (23) and, recalling that the random variables $\xi_N^\ell(\sigma)$ are statistically independent for different ℓ, we find after some long but straightforward computations,

$$-\frac{d}{dt}\frac{1}{N_K\beta}E\ln\tilde{Z}_{N_K}^{(ij)}(t,\beta)=$$

$$-\frac{\beta}{2}\left\langle f(Q_{N_K}(\sigma,\sigma))-\frac{1}{n}\sum_{\ell=1}^{n}f(Q_{N_{K-1}}(\sigma^{(\ell)},\sigma^{(\ell)}))\right\rangle \tag{33}$$

$$+\frac{\beta}{2}\left\langle f(Q_{N_K}(\sigma,\sigma'))-\frac{1}{n}\sum_{\ell=1}^{n}f(Q_{N_{K-1}}(\sigma^{(\ell)},\sigma'^{(\ell)}))\right\rangle \tag{34}$$

$$+\left\langle g(m_{N_K}(\sigma))-\frac{1}{n}\sum_{\ell=1}^{n}g(m_{N_{K-1}}(\sigma^{(\ell)}))\right\rangle+O\left(\frac{1}{N_K}\right), \tag{35}$$

where the averages are, of course, restricted to configurations belonging to $\tilde{C}_{ij}$. Since Q_{N_K} is a convex combination of the $Q_{N_{K-1}}$ and f is a convex function, the term (34) is not positive. On the other hand, since f is a function of class C^1, and $Q_{N_{K-1}}(\sigma^{(\ell)},\sigma^{(\ell)})$ and are constrained to belong to the interval $[i\varepsilon,(i+1)\varepsilon)$ for each ℓ, the term (33) is of order ε. The same holds for the term (35). This implies that, for K large enough,

$$-\frac{d}{dt}\frac{1}{N_K\beta}E\ln\tilde{Z}_{N_K}^{(ij)}(t,\beta)\leq C\varepsilon, \tag{36}$$

for some positive constant C independent of N. Recalling Eqs. (31), (32), this means that

$$-\frac{1}{N_K\beta}E\ln Z_{N_K}^{(ij)}(\beta)+\frac{1}{N_{K-1}\beta}E\ln Z_{N_{K-1}}^{(ij)}(\beta)\leq C\varepsilon. \tag{37}$$

Now, we want to turn this inequality, which involves disorder averages, into an inequality valid $J-$almost everywhere. To this purpose, we choose $\varepsilon=N_K^{-1/4}$ and we observe that, thanks to the estimate (17),

$$P\left(-\frac{1}{N_K\beta}\ln Z_{N_K}^{(ij)}(\beta)\geq-\frac{1}{N_K\beta}E\ln Z_{N_K}^{(ij)}(\beta)+C\varepsilon\right)\leq 2\exp\left(-\frac{\sqrt{N_K}C^2}{2}\right) \tag{38}$$

and

$$P\left(-\frac{1}{N_{K-1}\beta}\ln Z^{(ij)}_{N_{K-1}}(\beta)\leq -\frac{1}{N_{K-1}\beta}E\ln Z^{(ij)}_{N_{K-1}}(\beta)-C\varepsilon\right)\leq 2\exp\left(-\frac{\sqrt{N_K}C^2}{2n}\right). \quad (39)$$

Therefore, with probability $P\geq 1-4\sqrt{N_K}\exp\left(-\frac{\sqrt{N_K}C^2}{2n}\right)$, one has

$$-\frac{1}{N_K\beta}\ln Z^{(ij)}_{N_K}(\beta)\leq -\frac{1}{N_{K-1}\beta}\ln Z^{(ij)}_{N_{K-1}}(\beta)+3CN_K^{-1/4}\quad \forall i,j=0,\dots,[N_K^{1/4}]. \quad (40)$$

Since the probability of the complementary event is summable in K, it follows from Borel-Cantelli lemma [17] that inequality (40) holds J-almost surely, for K large enough. As a consequence, one obtains

$$\begin{aligned} Z_{N_K}(\beta) &= \sum_{i,j=0}^{[N_K^{1/4}]} Z^{(ij)}_{N_K}(\beta)\geq e^{-3\beta CN_K^{3/4}}\sum_{i,j=0}^{[N_K^{1/4}]}\left(Z^{(ij)}_{N_{K-1}}(\beta)\right)^n \quad (41)\\ &\geq e^{-3\beta CN_K^{3/4}}N_K^{(1-n)/2}\left(\sum_{i,j=0}^{[N_K^{1/4}]}Z^{(ij)}_{N_{K-1}}(\beta)\right)^n = \frac{e^{-3\beta CN_K^{3/4}}}{N_K^{(n-1)/2}}Z^n_{N_{K-1}}(\beta). \end{aligned}$$

Here, we have used the property

$$\sum_{i=1}^k x_i^n\geq k^{1-n}\left(\sum_{i=1}^k x_i\right)^n, \quad (42)$$

which holds if $x_i\geq 0$, thanks to the convexity of the function x^n. By taking the logarithm and dropping terms of lower order in N_K, one has

$$-\frac{1}{N_K\beta}\ln Z_{N_K}(\beta)\leq -\frac{1}{N_{K-1}\beta}\ln Z_{N_{K-1}}(\beta)+3CN_K^{-1/4}\quad J-a.s., \quad (43)$$

for K large enough. Notice that, with respect to (37), the above inequality involves the original free energy, where the sum over over configuration has no restrictions. From (43), it follows that the thermodynamic limit exists, J-almost surely, the term $N_K^{-1/4}$ being inessential. On the other hand, the exponential estimate (12), together with Borel-Cantelli Lemma, implies that the limit has a non random value $f(\beta)$, for almost every disorder realization J.

Once the almost sure convergence is proved, the convergence of the quenched average can be obtained easily, provided that the probability that $1/N\ln Z_N$ assumes large values is sufficiently small. For instance, one has the following

criterion [16]: given random variables X_K and X, if $X_K \longrightarrow X$ almost surely for $K \to \infty$, and if

$$\lim_{\lambda\to\infty} \sup_K E\left(|X_K|\, \mathcal{X}(\{|X_K| \geq \lambda\})\right) = 0, \tag{44}$$

where $\mathcal{X}(A)$ denotes the characteristic function of the set A, then

$$EX_K \to EX.$$

In the present case, $X_K = -(1/N_K\beta)\ln Z_{N_K}(\beta)$, $X = f(\beta)$, and the condition (44) can be easily checked, by employing the exponential bound (12).

In conclusion, we have proved almost sure convergence for the free energy, and convergence of its quenched average, for any subsequence of the form $\{N_0 n^K\}$. It is not difficult to show, by standard methods, that this implies convergence along any increasing subsequence $\{N_K\}$, and the uniqueness of the limit. $\square$

5 Conclusions

In this paper, we have extended the class of mean field spin glass models for which the existence of the thermodynamic limit can be proved, independently of an explicit calculation of the limit itself. Essentially, with respect to the models considered in [1], one abandons the assumption that the variance of the Hamiltonian is independent of the configuration, and that its mean value is additive in the size of the system. Some of the hypotheses of the Theorem, like the uniform bounds (5), (8), are required only for technical reasons, but can be dispensed with, at the expense of some extra work. On the other hand, the condition of convexity for the covariance function is essential, so that our result does not extend directly, for instance, to the p-spin model, with p odd.

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